

EXPLICIT FORMULAS FOR PARTITION PAIRS AND TRIPLES WITH 3-CORES

LIUQUAN WANG

ABSTRACT. Let $A_3(n)$ (resp. $B_3(n)$) denote the number of partition pairs (resp. triples) of n where each partition is 3-core. By applying Ramanujan's ${}_1\psi_1$ formula and Bailey's ${}_6\psi_6$ formula, we find the explicit formulas for $A_3(n)$ and $B_3(n)$. Using these formulas, we confirm a conjecture of Xia and establish many arithmetic identities satisfied by $A_3(n)$ and $B_3(n)$.

1. INTRODUCTION

A partition of a positive integer n is any nonincreasing sequence of positive integers whose sum is n . For example, $6 = 3 + 2 + 1$ and $\lambda = \{3, 2, 1\}$ is a partition of 6. A partition λ of n is said to be a t -core if it has no hook numbers that are multiples of t . We denote the number of t -core partitions of n by $a_t(n)$.

The generating function of $a_t(n)$ is given by (see [6, Eq. (2.1)])

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_{\infty}^t}{(q; q)_{\infty}}, \quad (1.1)$$

here and throughout this paper, we use the following notation

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad (a; q)_n := \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} \quad (-\infty < n < \infty).$$

For convenience, we also introduce the brief notation

$$(a_1, a_2, \dots, a_n; q)_{\infty} := (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_n; q)_{\infty}.$$

A partition k -tuple $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of n is a k -tuple of partitions $\lambda_1, \lambda_2, \dots, \lambda_k$ such that the sum of all the parts equals n . For example, let $\lambda_1 = \{2, 1\}$, $\lambda_2 = \{1, 1\}$, $\lambda_3 = \{1\}$. Then (λ_1, λ_2) is a partition pair of 5 since $2 + 1 + 1 + 1 = 5$, and $(\lambda_1, \lambda_2, \lambda_3)$ is a partition triple of 6 since $2 + 1 + 1 + 1 + 1 = 6$. A partition k -tuple of n with t -cores is a partition k -tuple $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of n where each λ_i is t -core for $i = 1, 2, \dots, k$.

Let $A_t(n)$ (resp. $B_t(n)$) denote the number of partition pairs (resp. triples) of n with t -cores. From (1.1) we know the generating functions for $A_t(n)$ and $B_t(n)$ are

$$\sum_{n=0}^{\infty} A_t(n) q^n = \frac{(q^t; q^t)_{\infty}^{2t}}{(q; q)_{\infty}^2} \quad (1.2)$$

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and

$$\sum_{n=0}^{\infty} B_t(n) q^n = \frac{(q^t; q^t)_{\infty}^{3t}}{(q; q)_{\infty}^3} \quad (1.3)$$

respectively.

In this paper, we focus on partition k -tuples with 3-cores for $1 \leq k \leq 3$. The properties of $a_3(n)$, $A_3(n)$ and $B_3(n)$ have drawn much attention in the past years. In 1996, using the tools of modular forms, Granville and Ono [8] first discovered the following formula for $a_3(n)$:

$$a_3(n) = d_{1,3}(3n+1) - d_{2,3}(3n+1), \quad (1.4)$$

where $d_{r,3}(n)$ denote the number of positive divisors of n congruent to r modulo 3.

In 2009, by using some known identities, Hirschhorn and Sellers [9] provided an elementary proof of (1.4). Moreover, let

$$3n+1 = \prod_{p_i \equiv 1 \pmod{3}} p_i^{\alpha_i} \cdot \prod_{q_j \equiv 2 \pmod{3}} q_j^{\beta_j}$$

with each $\alpha_i, \beta_j \geq 0$ be the prime factorization of $3n+1$, they gave the explicit formula:

$$a_3(n) = \begin{cases} \prod (\alpha_i + 1) & \text{if all } \beta_j \text{ are even;} \\ 0 & \text{otherwise.} \end{cases}$$

Some arithmetic identities were then obtained as corollaries. For example, let $p \equiv 2 \pmod{3}$ be a prime and let k be a positive even integer. Then, for all $n \geq 0$,

$$a_3\left(p^k n + \frac{p^k - 1}{3}\right) = a_3(n).$$

In 2014, Lin [10] found some arithmetic relations about $A_3(n)$ such as $A_3(8n+6) = 7A_3(2n+1)$. By using some theta function identities, Baruah and Nath [4] established three infinite families of arithmetic identities involving $A_3(n)$. For any integer $k \geq 1$, they proved that

$$\begin{aligned} A_3\left(2^{2k+2}n + \frac{2(2^{2k} - 1)}{3}\right) &= \frac{2^{2k+2} - 1}{3} A_3(4n), \\ A_3\left(2^{2k+2}n + \frac{2(2^{2k+2} - 1)}{3}\right) &= \frac{2^{2k+2} - 1}{3} \cdot A_3(4n+2) - \frac{2^{2k+2} - 4}{3} \cdot A_3(n), \quad (1.5) \\ A_3\left(2^{2k+1}n + \frac{5 \cdot 2^{2k} - 2}{3}\right) &= (2^{2k+1} - 1) A_3(2n+1). \end{aligned}$$

Xia [12] found several infinite families of congruences modulo 4, 8 for $A_3(n)$. For example, he showed that for all integers $n \geq 0$,

$$A_3(8n+4) \equiv 0 \pmod{4}, \quad A_3(16n+4) \equiv 0 \pmod{8}.$$

He also proposed the following conjecture

Conjecture 1.1. *For any positive integer j and prime p , there exists a positive integer k_0 such that for all $n \geq 0$ and $\alpha \geq 0$,*

$$A_3\left(4^{k_0(\alpha+1)}n + \frac{2^{2k_0(\alpha+1)-1} - 2}{3}\right) \equiv 0 \pmod{p^j}.$$

For more results about $a_3(n)$ and $A_3(n)$, see [1]–[4] and [9, 10, 12, 13].

Recently, the author [11] studied the arithmetic properties of $B_3(n)$. By elementary q series manipulations, we found three infinite families of arithmetic identities satisfied by $B_3(n)$. For any integer $k \geq 1$, we proved that

$$B_3(3^k n + 3^k - 1) = 3^{2k} B_3(n), \quad (1.6)$$

$$B_3(2^{k+1} n + 2^k - 1) = \frac{2^{2k+2} + (-1)^k}{5} \cdot B_3(2n), \quad (1.7)$$

and

$$B_3(2^{k+1} n + 2^{k+1} - 1) = \frac{2^{2k+2} + (-1)^k}{5} \cdot B_3(2n+1) + \frac{2^{2k+2} - 4(-1)^k}{5} \cdot B_3(n). \quad (1.8)$$

In viewing of (1.4), it is natural to ask that whether we can find the explicit formulas for $A_3(n)$ and $B_3(n)$ or not. In this paper, we give a positive answer to this question. By using Ramanujan's ${}_1\psi_1$ summation formula and Bailey's ${}_6\psi_6$ formula, we give a new simple proof of (1.4) and find the explicit formulas for $A_3(n)$ and $B_3(n)$. With these formulas in mind, most of the results mentioned above become direct consequences. In particular, we will confirm Conjecture 1.1 and give some generalizations of (1.5)–(1.8).

2. EXPLICIT FORMULA FOR $A_3(n)$

Before we present the explicit formula for $A_3(n)$, we provide a new elementary proof of (1.4). The key tool in this section is Ramanujan's ${}_1\psi_1$ summation formula [5, Theorem 1.3.12].

Lemma 2.1 (Ramanujan's ${}_1\psi_1$ Summation). *For $|b/a| < |z| < 1$ and $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(az, q/(az), q, b/a; q)_{\infty}}{(z, b/(az), b, q/a; q)_{\infty}}. \quad (2.1)$$

Proof of (1.4). Setting $t = 3$ in (1.1), we get

$$\sum_{n=0}^{\infty} a_3(n) q^n = \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}}.$$

Note that

$$(q; q)_{\infty} = (q; q^3)_{\infty} (q^2; q^3)_{\infty} (q^3; q^3)_{\infty}, \quad (2.2)$$

we have

$$\sum_{n=0}^{\infty} a_3(n) q^n = \frac{(q^3; q^3)_{\infty}^2}{(q; q^3)_{\infty} (q^2; q^3)_{\infty}}. \quad (2.3)$$

Taking $(a, b, z, q) \rightarrow (q, q^4, q, q^3)$ in (2.1), we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q; q^3)_n}{(q^4; q^3)_n} \cdot q^n = \frac{(q^2, q, q^3, q^3; q^3)_{\infty}}{(q, q^2, q^4, q^2; q^3)_{\infty}}.$$

Dividing both sides by $1 - q$, after simplification, we get

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{3n+1}} = \frac{(q^3; q^3)_{\infty}^2}{(q; q^3)_{\infty} (q^2; q^3)_{\infty}}. \quad (2.4)$$

Combining (2.3) with (2.4), we obtain

$$\sum_{n=0}^{\infty} a_3(n)q^n = \sum_{n=-\infty}^{\infty} \frac{q^n}{1 - q^{3n+1}}.$$

Replacing q by q^3 and multiplying both sides by q , we get

$$\begin{aligned} \sum_{n=0}^{\infty} a_3(n)q^{3n+1} &= \sum_{m=0}^{\infty} \frac{q^{3m+1}}{1 - q^{3(3m+1)}} + \sum_{m=-\infty}^{-1} \frac{q^{3m+1}}{1 - q^{3(3m+1)}} \\ &= \sum_{m=0}^{\infty} \frac{q^{3m+1}}{1 - q^{3(3m+1)}} + \sum_{m=0}^{\infty} \frac{q^{-3m-2}}{1 - q^{3(-3m-2)}} \\ &= \sum_{m=0}^{\infty} \frac{q^{3m+1}}{1 - q^{3(3m+1)}} - \sum_{m=0}^{\infty} \frac{q^{2(3m+2)}}{1 - q^{3(3m+2)}} \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} q^{(3m+1)(3k+1)} - \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} q^{(3m+2)(3k+2)}, \end{aligned} \quad (2.5)$$

here the second equality follows by replacing m by $-m-1$ in the second summation.

Now (1.4) follows by comparing the coefficients of q^{3n+1} on both sides of (2.5). \square

Let $\sigma(n)$ denote the sum of positive divisors of n . Applying the method in proving (1.4), we can find the explicit formula for $A_3(n)$.

Theorem 2.1. *For any integer $n \geq 0$, we have $A_3(n) = \frac{1}{3}\sigma(3n+2)$. If we write $3n+2 = \prod_{i=1}^s p_i^{\alpha_i}$ as the unique prime factorization, then*

$$A_3(n) = \frac{1}{3} \prod_{i=1}^s \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

Proof. Setting $t = 3$ in (1.2), and applying (2.2) we obtain that

$$\sum_{n=0}^{\infty} A_3(n)q^n = \frac{(q^3; q^3)_{\infty}^4}{(q; q^3)_{\infty}^2 (q^2; q^3)_{\infty}^2}. \quad (2.6)$$

Taking $(a, b, q) \rightarrow (q, q^4, q^3)$ in (2.1), and dividing both sides by $1 - \frac{q^2}{z}$, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(q; q^3)_n}{(q^4; q^3)_n} \cdot \frac{z^n}{1 - q^2/z} = \frac{(qz, q^5/z, q^3, q^3; q^3)_{\infty}}{(z, q^3/z, q^4, q^2; q^3)_{\infty}}. \quad (2.7)$$

Let $z \rightarrow q^2$. By L'Hospital's rule, we deduce that

$$\sum_{n=-\infty}^{\infty} \frac{(1-q) \cdot nq^{2n}}{1 - q^{3n+1}} = \frac{(q^3; q^3)_{\infty}^4}{(q^2; q^3)_{\infty}^2 (q; q^3)_{\infty} (q^4; q^3)_{\infty}}.$$

Dividing both sides by $1 - q$ and combining with (2.6), we obtain

$$\sum_{n=0}^{\infty} A_3(n)q^n = \frac{(q^3; q^3)_{\infty}^4}{(q; q^3)_{\infty}^2 (q^2; q^3)_{\infty}^2} = \sum_{n=-\infty}^{\infty} \frac{nq^{2n}}{1 - q^{3n+1}}. \quad (2.8)$$

Replacing q by q^3 and multiplying both sides by q^2 , we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} A_3(n) q^{3n+2} &= \sum_{m=0}^{\infty} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} + \sum_{m=-\infty}^{-1} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} \\
&= \sum_{m=0}^{\infty} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} + \sum_{m=0}^{\infty} \frac{(-m-1)q^{2(-3m-2)}}{1-q^{3(-3m-2)}} \\
&= \sum_{m=0}^{\infty} \frac{mq^{2(3m+1)}}{1-q^{3(3m+1)}} + \sum_{m=0}^{\infty} \frac{(m+1)q^{3m+2}}{1-q^{3(3m+2)}} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} mq^{(3m+1)(3k+2)} + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (m+1)q^{(3m+2)(3k+1)} \\
&= \frac{1}{3} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left((3m+1)q^{(3m+1)(3k+2)} + (3m+2)q^{(3m+2)(3k+1)} \right) \\
&\quad + \frac{1}{3} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left(q^{(3m+2)(3k+1)} - q^{(3m+1)(3k+2)} \right)
\end{aligned} \tag{2.9}$$

Interchanging the roles of k and m , we see that

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} q^{(3m+2)(3k+1)} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} q^{(3k+2)(3m+1)} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} q^{(3m+1)(3k+2)}.$$

Thus the second sum in the right hand side of (2.9) vanishes. Comparing the coefficients of q^{3n+2} on both sides of (2.9), we prove the first assertion of the theorem. The second assertion then follows immediately. \square

Once we know the explicit formula of $A_3(n)$, we can verify those identities in (1.5) by simple arguments. For example, since $\sigma(n)$ is multiplicative, by Theorem 2.1 we have

$$A_3(4n) = \frac{1}{3} \sigma(2(6n+1)) = \frac{1}{3} \sigma(2) \sigma(6n+1) = \sigma(6n+1),$$

$$A_3\left(2^{2k+2}n + \frac{2(2^{2k}-1)}{3}\right) = \frac{1}{3} \sigma(2^{2k+1}(6n+1)) = \frac{1}{3} \sigma(2^{2k+1}) \sigma(6n+1).$$

Note that $\sigma(2^{2k+1}) = 2^{2k+2} - 1$, this proves the first identity in (1.5). Others can be proved in a similar way.

Moreover, we can extend (1.5) to some large families of arithmetic identities.

Theorem 2.2. *Let p be a prime and k, n be nonnegative integers.*

(1) *If $p \equiv 1 \pmod{3}$, we have*

$$A_3\left(p^k n + \frac{2p^k - 2}{3}\right) = \frac{p^k - 1}{p - 1} A_3\left(pn + \frac{2p - 2}{3}\right) - \frac{p^k - p}{p - 1} A_3(n).$$

(2) *If $p \equiv 2 \pmod{3}$, we have*

$$A_3\left(p^{2k} n + \frac{2p^{2k} - 2}{3}\right) = \frac{p^{2k} - 1}{p^2 - 1} A_3\left(p^2 n + \frac{2p^2 - 2}{3}\right) - \frac{p^{2k} - p^2}{p^2 - 1} A_3(n).$$

Proof. We write $3n + 2 = p^m N$, where N is an integer not divisible by p .

(1) By Theorem 2.1 we deduce that

$$A_3(n) = \frac{1}{3}\sigma(p^m N) = \frac{1}{3}\sigma(p^m)\sigma(N) = \frac{1}{3} \cdot \frac{p^{m+1} - 1}{p - 1}\sigma(N), \quad (2.10)$$

Similarly we have

$$A_3\left(pn + \frac{2p - 2}{3}\right) = \frac{1}{3}\sigma(p^{m+1}N) = \frac{1}{3} \cdot \frac{p^{m+2} - 1}{p - 1}\sigma(N), \quad (2.11)$$

$$A_3\left(p^k n + \frac{2p^k - 2}{3}\right) = \frac{1}{3}\sigma(p^{k+m}N) = \frac{1}{3} \cdot \frac{p^{k+m+1} - 1}{p - 1}\sigma(N). \quad (2.12)$$

Now the assertion follows from (2.10)–(2.12) by direct verification.

(2) In the same way we have

$$A_3\left(p^2 n + \frac{2p^2 - 2}{3}\right) = \frac{1}{3}\sigma(p^{m+2}N) = \frac{1}{3} \cdot \frac{p^{m+3} - 1}{p - 1}\sigma(N), \quad (2.13)$$

and

$$A_3\left(p^{2k} n + \frac{2p^{2k} - 2}{3}\right) = \frac{1}{3}\sigma(p^{2k+m}N) = \frac{1}{3} \cdot \frac{p^{2k+m+1} - 1}{p - 1}\sigma(N). \quad (2.14)$$

Combining (2.10), (2.13) and (2.14), we prove the assertion by direct verification. \square

Setting $p = 2, 5, 7$ in Theorem 2.3, we obtain the following arithmetic identities for $k, n \geq 0$,

$$\begin{aligned} A_3\left(2^{2k}n + \frac{2^{2k+1} - 2}{3}\right) &= \frac{2^{2k} - 1}{3}A_3(4n + 2) - \frac{2^{2k} - 4}{3}A_3(n), \\ A_3\left(5^{2k}n + \frac{2 \cdot 5^{2k} - 2}{3}\right) &= \frac{5^{2k} - 1}{24}A_3(25n + 16) - \frac{5^{2k} - 25}{24}A_3(n), \\ A_3\left(7^k n + \frac{2 \cdot 7^k - 2}{3}\right) &= \frac{7^k - 1}{6}A_3(7n + 4) - \frac{7^k - 7}{6}A_3(n). \end{aligned}$$

Theorem 2.3. Let p be a prime and k, n be nonnegative integers such that $p \nmid 3n + 2$.

(1) If $p \equiv 1 \pmod{3}$, we have

$$A_3\left(p^k n + \frac{2p^k - 2}{3}\right) = \frac{p^{k+1} - 1}{p - 1}A_3(n).$$

(2) If $p \equiv 2 \pmod{3}$, we have

$$A_3\left(p^{2k} n + \frac{2p^{2k} - 2}{3}\right) = \frac{p^{2k+1} - 1}{p - 1}A_3(n).$$

Proof. From Theorem 2.1, we deduce that

$$A_3\left(p^k n + \frac{2p^k - 2}{3}\right) = \frac{1}{3}\sigma(p^k(3n + 2)) = \frac{1}{3}\sigma(p^k)\sigma(3n + 2) = \frac{p^{k+1} - 1}{p - 1}A_3(n).$$

This implies (1). (2) can be proved in a similar way. \square

For example, let $p = 2$ and replacing n by $2n + 1$ in (2), we obtain the third identity of (1.5). If we set $p = 5$ (resp. $p = 7$) and replace n by $5n + r$ (resp. $7n + r$), we deduce that for $k, n \geq 0$,

$$A_3\left(5^{2k}(5n + r) + \frac{2 \cdot 5^{2k} - 2}{3}\right) = \frac{5^{2k+1} - 1}{4}A_3(5n + r), \quad r \in \{0, 2, 3, 4\}$$

and

$$A_3\left(7^k(7n+r) + \frac{2 \cdot 7^k - 2}{3}\right) = \frac{7^{k+1} - 1}{6} A_3(7n+r), \quad r \in \{0, 1, 2, 3, 5, 6\}.$$

We conclude this section by proving Conjecture 1.1.

Proof of Conjecture 1.1. By Theorem 2.1, we get

$$\begin{aligned} A_3\left(4^{k_0(\alpha+1)}n + \frac{2^{2k_0(\alpha+1)-1} - 2}{3}\right) &= \frac{1}{3} \sigma(2^{2k_0(\alpha+1)-1}(6n+1)) \\ &= \frac{2^{2k_0(\alpha+1)} - 1}{3} \sigma(6n+1). \end{aligned} \quad (2.15)$$

Let $k_0 = \frac{1}{2}p^j(p-1)$. Since $2k_0(\alpha+1) \equiv 0 \pmod{p^j(p-1)}$, by Euler's theorem, we have $2^{2k_0(\alpha+1)} \equiv 1 \pmod{p^{j+1}}$. From (2.15) the conjecture follows immediately. \square

Indeed, most of the congruences found by Xia [12] can be proved by using Theorem 2.1. We omit the details here.

3. EXPLICIT FORMULA FOR $B_3(n)$

In order to find the explicit formula for $B_3(n)$, we need the following formula.

Lemma 3.1 (Bailey's ${}_6\psi_6$ formula). *For $|qa^2/(bcde)| < 1$,*

$$\begin{aligned} &{}_6\psi_6\left(\frac{q\sqrt{a}, -q\sqrt{a}, b, c, d, e}{\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e}; q, \frac{qa^2}{bcde}\right) \\ &= \frac{(aq, aq/(bc), aq/(bd), aq/(be), aq/(cd), aq/(ce), aq/(de), q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/(bcde); q)_\infty}. \end{aligned} \quad (3.1)$$

For the proof of this lemma, see [7, Sec. 5.3].

Theorem 3.1. *For any integer $n \geq 0$, we have*

$$B_3(n) = \sum_{\substack{d|n+1 \\ d \equiv 1 \pmod{3}}} \left(\frac{n+1}{d}\right)^2 - \sum_{\substack{d|n+1 \\ d \equiv 2 \pmod{3}}} \left(\frac{n+1}{d}\right)^2.$$

Furthermore, if we write

$$n+1 = 3^\alpha \prod_{p_i \equiv 1 \pmod{3}} p_i^{\alpha_i} \prod_{q_j \equiv 2 \pmod{3}} q_j^{\beta_j}$$

as the unique prime factorization of $n+1$ with $\alpha, \alpha_i, \beta_j \geq 0$, then

$$B_3(n) = 3^{2\alpha} \prod_{p_i \equiv 1 \pmod{3}} \frac{p_i^{2(\alpha_i+1)} - 1}{p_i^2 - 1} \prod_{q_j \equiv 2 \pmod{3}} \frac{q_j^{2\beta_j+2} + (-1)^{\beta_j}}{q_j^2 + 1}.$$

Proof. Setting $t = 3$ in (1.3) and applying (2.2), we see that

$$\sum_{n=0}^{\infty} B_3(n) q^n = \frac{(q^3; q^3)_\infty^6}{(q; q^3)_\infty^3 (q^2; q^3)_\infty^3}. \quad (3.2)$$

Taking $(a, b, c, d, e, q) \rightarrow (q^2, q, q, q, q, q^3)$ in (3.1), then multiplying both sides by $\frac{q(1-q^2)}{(1-q)^4}$, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{(1+q^{3n+1})q^{3n+1}}{(1-q^{3n+1})^3} = q \cdot \frac{(q^3; q^3)_\infty^6}{(q; q^3)_\infty^3 (q^2; q^3)_\infty^3}.$$

Combining this with (3.2), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} B_3(n)q^{n+1} &= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{3m+1})}{(1-q^{3m+1})^3} + \sum_{m=-\infty}^{-1} \frac{q^{3m+1}(1+q^{3m+1})}{(1-q^{3m+1})^3} \\ &= \sum_{m=0}^{\infty} \frac{q^{3m+1}(1+q^{3m+1})}{(1-q^{3m+1})^3} - \sum_{m=0}^{\infty} \frac{q^{3m+2}(1+q^{3m+2})}{(1-q^{3m+2})^3}, \end{aligned} \quad (3.3)$$

here the second equality follows by replacing m by $-m-1$ in the second sum.

It is well known that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1.$$

Applying the operator $x \frac{d}{dx}$ twice to both sides, we get

$$\frac{x(1+x)}{(1-x)^3} = \sum_{k=1}^{\infty} k^2 x^k, \quad |x| < 1.$$

Applying this identity to (3.3), we obtain

$$\sum_{n=0}^{\infty} B_3(n)q^{n+1} = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} k^2 (q^{(3m+1)k} - q^{(3m+2)k}).$$

The first assertion of this theorem now follows immediately by comparing the coefficients of q^{n+1} on both sides.

Let

$$f(n) = \sum_{\substack{d|n \\ d \equiv 1 \pmod{3}}} \left(\frac{n}{d}\right)^2 - \sum_{\substack{d|n \\ d \equiv 2 \pmod{3}}} \left(\frac{n}{d}\right)^2.$$

Suppose m and n are integers which are coprime to each other. It is not hard to see that

$$\begin{aligned} f(mn) &= \sum_{\substack{d|mn \\ d \equiv 1 \pmod{3}}} \left(\frac{mn}{d}\right)^2 - \sum_{\substack{d|mn \\ d \equiv 2 \pmod{3}}} \left(\frac{mn}{d}\right)^2 \\ &= \sum_{\substack{d_1|m \\ d_1 \equiv 1 \pmod{3}}} \sum_{\substack{d_2|n \\ d_2 \equiv 1 \pmod{3}}} + \sum_{\substack{d_1|m \\ d_1 \equiv 2 \pmod{3}}} \sum_{\substack{d_2|n \\ d_2 \equiv 2 \pmod{3}}} \left(\frac{mn}{d_1 d_2}\right)^2 \\ &\quad - \sum_{\substack{d_1|m \\ d_1 \equiv 1 \pmod{3}}} \sum_{\substack{d_2|n \\ d_2 \equiv 2 \pmod{3}}} - \sum_{\substack{d_1|m \\ d_1 \equiv 2 \pmod{3}}} \sum_{\substack{d_2|n \\ d_2 \equiv 1 \pmod{3}}} \left(\frac{mn}{d_1 d_2}\right)^2 \\ &= \left(\sum_{\substack{d_1|m \\ d_1 \equiv 1 \pmod{3}}} \left(\frac{m}{d_1}\right)^2 - \sum_{\substack{d_1|m \\ d_1 \equiv 2 \pmod{3}}} \left(\frac{m}{d_1}\right)^2 \right) \\ &\quad \cdot \left(\sum_{\substack{d_2|n \\ d_2 \equiv 1 \pmod{3}}} \left(\frac{n}{d_2}\right)^2 - \sum_{\substack{d_2|n \\ d_2 \equiv 2 \pmod{3}}} \left(\frac{n}{d_2}\right)^2 \right) \\ &= f(m)f(n) \end{aligned}$$

This implies that $f(n)$ is multiplicative. For any prime p , from the definition of $f(n)$ and by direct calculations, we obtain that

$$f(p^k) = \begin{cases} 3^{2k} & \text{if } p = 3 \\ \frac{p^{2(k+1)} - 1}{p^2 - 1} & \text{if } p \equiv 1 \pmod{3} \\ \frac{p^{2k+2} + (-1)^k}{p^2 + 1} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

The second assertion of this theorem then follows since $f(n)$ is multiplicative and $B_3(n) = f(n+1)$. \square

Theorem 3.2. *Let p be a prime and k, n be nonnegative integers.*

(1) *If $p \equiv 1 \pmod{3}$, we have*

$$B_3(p^k n + p^k - 1) = \frac{p^{2k} - 1}{p^2 - 1} B_3(pn + p - 1) - \frac{p^{2k} - p^2}{p^2 - 1} B_3(n).$$

(2) *If $p \equiv 2 \pmod{3}$, we have*

$$B_3(p^k n + p^k - 1) = \frac{p^{2k} - (-1)^k}{p^2 + 1} B_3(pn + p - 1) + \frac{p^{2k} + (-1)^k p^2}{p^2 + 1} B_3(n).$$

Proof. Let $n+1 = p^m N$, where N is not divisible by p .

(1) Since $f(n)$ is multiplicative, we have

$$\begin{aligned} B_3(n) &= f(n+1) = f(p^m) f(N) = \frac{p^{2(m+1)} - 1}{p^2 - 1} f(N), \\ B_3(pn + p - 1) &= f(p(n+1)) = f(p^{m+1}) f(N) = \frac{p^{2(m+2)} - 1}{p^2 - 1} f(N), \\ B_3(p^k n + p^k - 1) &= f(p^k(n+1)) = f(p^{k+m}) f(N) = \frac{p^{2(m+k+1)} - 1}{p^2 - 1} f(N). \end{aligned} \quad (3.4)$$

From those identities in (3.4), we prove (1) by direct verification.

(2) Similarly, we have

$$\begin{aligned} B_3(n) &= f(n+1) = f(p^m) f(N) = \frac{p^{2(m+1)} + (-1)^m}{p^2 + 1} f(N), \\ B_3(pn + p - 1) &= f(p(n+1)) = f(p^{m+1}) f(N) = \frac{p^{2(m+2)} + (-1)^{m+1}}{p^2 + 1} f(N), \\ B_3(p^k n + p^k - 1) &= f(p^k(n+1)) = f(p^{k+m}) f(N) = \frac{p^{2(m+k+1)} + (-1)^{m+k}}{p^2 + 1} f(N). \end{aligned} \quad (3.5)$$

From those identities in (3.5), we prove (2) by direct verification. \square

By setting $p = 2$ in this theorem we obtain (1.8) immediately. For more examples, by setting $p = 5, 7$ in this theorem, we obtain for $k, n \geq 0$,

$$B_3(5^k n + 5^k - 1) = \frac{5^{2k} - (-1)^k}{26} B_3(5n + 4) + \frac{5^{2k} + 25(-1)^k}{26} B_3(n)$$

and

$$B_3(7^k n + 7^k - 1) = \frac{7^{2k} - 1}{48} B_3(7n + 6) - \frac{7^{2k} - 49}{48} B_3(n).$$

In some special cases, we can obtain some relations between $B_3(p^k n + p^k - 1)$ and $B_3(n)$.

Theorem 3.3. *Let p be a prime and k, n be nonnegative integers.*

(1) *If $p = 3$, we have $B_3(3^k n + 3^k - 1) = 3^{2k} B_3(n)$.*

(2) *If $p \equiv 1 \pmod{3}$ and $p \nmid n + 1$, then*

$$B_3(p^k n + p^k - 1) = \frac{p^{2(k+1)} - 1}{p^2 - 1} B_3(n).$$

(3) *If $p \equiv 2 \pmod{3}$ and $p \nmid n + 1$, then*

$$B_3(p^k n + p^k - 1) = \frac{p^{2(k+1)} + (-1)^k}{p^2 + 1} B_3(n).$$

Proof. Let $n + 1 = p^m N$, where N is not divisible by p . By Theorem 3.1 and the fact that $f(n)$ is multiplicative, we get

$$B_3(n) = f(p^m N) = f(p^m) f(N)$$

and

$$B_3(p^k n + p^k - 1) = f(p^{k+m} N) = f(p^{k+m}) f(N).$$

(1) We have

$$B_3(3^k n + 3^k - 1) = 3^{2k+2m} f(N) = 3^{2k} B_3(n).$$

(2) Since $p \nmid n + 1$, we have $m = 0$ and

$$B_3(p^k n + p^k - 1) = f(p^k) f(N) = \frac{p^{2(k+1)} - 1}{p^2 - 1} B_3(n).$$

(3) Since $p \nmid n + 1$, we have $m = 0$ and

$$B_3(p^k n + p^k - 1) = f(p^k) f(N) = \frac{p^{2(k+1)} + (-1)^k}{p^2 + 1} B_3(n).$$

□

Note that in this theorem, (1) is (1.6) exactly. By setting $p = 2$ and replacing n by $2n$ in (3), we obtain (1.7) at once. For more examples, by setting $p = 5$ (resp. $p = 7$) and replacing n by $5n + r$ (resp. $7n + r$) in (3) (resp. (2)) we obtain for $k, n \geq 0$,

$$B_3(5^{k+1} n + 5^k(r + 1) - 1) = \frac{5^{2k+2} + (-1)^k}{26} B_3(5n + r), \quad r \in \{0, 1, 2, 3\}$$

and

$$B_3(7^{k+1} n + 7^k(r + 1) - 1) = \frac{7^{2k+2} - 1}{48} B_3(7n + r), \quad r \in \{0, 1, 2, 3, 4, 5\}$$

respectively.

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE, 119076, SINGAPORE

E-mail address: wangliuquan@u.nus.edu; mathlqwang@163.com